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by M. S. Natanzon

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FORCED DISCONTINUOUS OSCILLATIONS OF A FLUID IN A PIPELINE

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M.S.Natanzon

Development of obtaining asymptotic solutions of the problem of forced discontinuous cavitational oscillations of a fluid in a pipeline. It is shown that using these solutions it is possible to analyze the basic characteristics of discontinuous cavitational oscillations in the absence of energy dissipation, as well as to study the modes of discontinuous oscillations taking energy dissipation into account.

The flow of a liquid in a pipeline after reflection of the wave of a hydraulic ram (water-hammer) from the ends of this duct is often accompanied by the appearance of a region of negative pressure. The cavitation phenomena, occurring under certain conditions in this region, may lead to a substantial change in the character of motion of the fluid.

As an example of a flow of this kind, let us consider, in their most general features, the free oscillations of a fluid in a system consisting of a tank in which a constant pressure and level is maintained, and an elastic pipeline closed at one end (Fig.1).

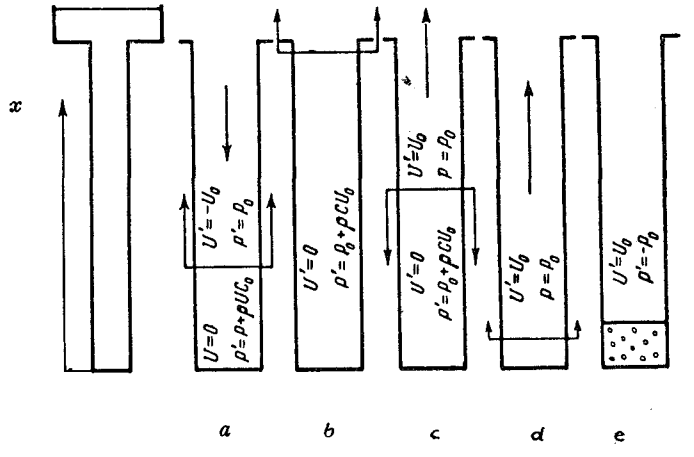

Fig.1

During a certain time interval, let the liquid column fall, under the action of gravity and of the pressure gradient between its upper and lower end. At a certain time ($t = 0$), let the liquid column reach the lowest end of the tube at a velocity of u_0, after which a hydraulic shock takes place. As a

* Numbers in the margin indicate pagination in the original foreign text.

result of this shock, a pressure wave moves upward from the lower end of the tube (Fig.1a). The velocity of the liquid behind the wave front will be zero, while ahead of the wave front it will have a value equal to the initial velocity of the liquid, u_0. After reflection of this wave from the upper end of the pipe, a rarefaction wave will travel downward (Fig.1c). Ahead of the wave front, the compressed liquid will be at rest; behind the wave front, the velocity of the fluid will be directed upward. After reflection of the wave from the lower end of the pipe, the velocity of the liquid in the immediate vicinity of the sidewall of the pipe will become zero, while the pressure will drop steeply. If the resultant low pressure is less than a certain critical value, a cavitational discontinuity will occur in the liquid, and the pressure on the closed end will take a value equal to the saturated vapor pressure of the liquid, p_s. Since, on the other hand, the velocity of the fluid at this time is directed upward, the liquid column detached from the lower cross section will be displaced as a whole, increasing the volume filled with a mixture of vapor bubbles and liquid. The force of gravity, as well as the pressure difference acting on the liquid column, will retard this motion. After the velocity of the liquid /34 column has reached zero, it will begin to fall. The fall of the liquid will end in a hydraulic shock. If none of the above processes are accompanied by energy dissipation and if some additional condition is satisfied (which we will consider later in the text), then the system will return to its initial state ($t = 0$). It is obvious from the above description that, in the system shown in Fig.1, under certain conditions, undamped free oscillations of a predominantly anharmonic character may be generated (hydraulic shocks of alternating regions of constant pressure p_s). This type of oscillation will hereafter be termed "discontinuous cavitational oscillations".

The problems connected with the motion of a fluid, created after occurrence of a cavitational discontinuity due to the reflections of the wave of a hydraulic shock, have been considered by various authors (Bibl.1 - 4).

However, an application of their methods to problems in investigations on forced discontinuous cavitational oscillations is ineffective. This is due to the fact that the exact solution of the problem generally does not lead to strictly periodic solutions and that already after a few periods the corrections necessary because of the deviation from strict periodicity become so extremely complex that they practically exclude any analyses of a general nature (Bibl.3).

In this report, we will describe a method of constructing, in a certain sense, asymptotic solutions of the problem of forced oscillations. These solutions permit a rather detailed analysis of the fundamental features of discontinuous cavitational oscillations in the absence of energy dissipation and provide the necessary prerequisites for investigating the modes of discontinuous oscillations, taking account of energy dissipation.

1. Formulation of the Problem

We will consider the forced oscillations of a liquid column bounded on the upper end by a tank with constant level and pressure and on the lower, by a piston performing the harmonic oscillations:

$$l_e = a \sin(\omega t + \varphi) \,. \tag{1.1}$$

Here l_e is the coordinate of the piston, a the oscillation amplitude of the piston, ω the angular velocity of the oscillations, φ the oscillation phase, at the time t = 0. In all the calculations given here, we will disregard the energy dissipation.

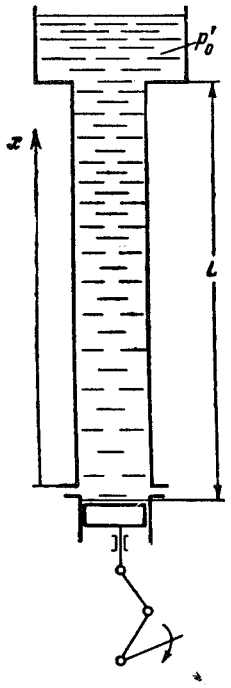

Fig.2

Figure 2 shows the computational scheme of the system. At a = 0, the problem changes to the problem of free oscillations of a liquid, one of whose possible solutions corresponds to the above-described mode of free discontinuous cavitational oscillations.

Using the assumptions conventional for problems of this kind, we will write the equations describing the motion of the compressed fluid in an elastic pipe in the following form:

$$\begin{aligned} \frac{\partial p'(x,t)}{\partial x} + \rho g &= -\rho \frac{\partial u(x,t)}{\partial t} \\ \frac{\partial p'(x,t)}{\partial t} &= -\rho c^2 \frac{\partial u(x,t)}{\partial x} \,. \end{aligned} \tag{1.2}$$

Here u is the velocity of the fluid, p' the pressure, ρ the density of the liquid, g the acceleration of gravity, c the speed of sound in the fluid (taking the effect of the elasticity of the walls into consideration).

The presence of a constant term (ρg) in the first equation of the

system (1.2) is due to the action of gravity.

Utilizing the substitution

$$p = p' + g\rho x \tag{1.3}$$

let us write eqs.(1.2) in a somewhat more convenient form: /35

$$\frac{\partial p}{\partial x} = -\rho \frac{\partial u}{\partial t}, \qquad \frac{\partial p}{\partial t} = -\rho c^2 \frac{\partial u}{\partial x} \tag{1.4}$$

On the basis of the new variables, the value of the quantity p, which we designate here as "pressure", will be constant along the liquid column* under stationary conditions, whereas eqs.(1.4) will have the same form as in the absence of gravitation. Accordingly, we can confine ourselves to a consideration of the problem that disregards gravitational forces. In this case, eq.(1.3) permits passing to the problem that takes the effect of gravity into consideration.

It is known that the general solution of eqs.(1.4) has the following form:

$$p = \psi_1(x - ct) + \psi_2(x + ct), \qquad \rho cu = \psi_1(x - ct) - \psi_2(x + ct). \tag{1.5}$$

Here ψ_1 and ψ_2 are two arbitrary functions whose specific form is determined by the boundary conditions and by the initial conditions of the problem. If there are no cavitational phenomena at the piston, then the use of the boundary conditions formulated at the beginning of this Section, together with eqs.(1.5), will yield the following well-known relation [cf., for example, (Bibl.5)], between the frequency and amplitude of the pressure oscillations at the piston, corresponding to the mode of acoustic oscillations:

$$\delta p = a\rho\omega \; c\varphi t \frac{\omega L}{c}. \tag{1.6}$$

Here δp is the amplitude of the pressure oscillations at the piston, and L the length of the pipeline. For convenience in our further discussion, it is useful to represent eqs.(1.6) in the following form:

$$\mu = 1 + 2\pi f^\circ \frac{\rho c^2}{p_0 - p_s} \frac{a}{L} \tan 2\pi f^\circ. \tag{1.7}$$

Here, μ and f° are the dimensionless amplitude of the pressure oscillations and the dimensionless frequency, defined respectively by the relations:

$$\mu = \frac{p_m - p_s}{p_0 - p_s}, \qquad f^\circ = \frac{\omega L}{2\pi c}. \tag{1.8}$$

Here, p_0 is the pressure in the liquid in the absence of oscillations (pressure in the tank), p_s the saturated vapor pressure of the liquid, p_m the maximum pressure arising in the liquid during the oscillations.

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* Accordingly, the true value of the pressure in the tank p_0' will be connected with p_0 by the relation $p_0' = p_0 - g\rho L$.

If the amplitude of the pressure oscillations in the liquid exceeds a certain critical value, which has been shown by experiment to be close to the value of $p_0 - p_s$, periodic discontinuities will be created in the liquid, and discontinuous cavitational oscillations will be established in the system. The values of the system parameters that are critical in this sense can be approximately determined by setting $\mu = 0$ in eq.(1.7).

It follows from eq.(1.7) that, in the region of frequencies up to the first resonance maximum, it is necessary to have a sufficiently high frequency or amplitude of the piston oscillations (the last condition is valid over the entire frequency range), in order for any cavitational phenomena to appear. However, it does not follow from this that in a region of frequencies and amplitudes below the critical values, acoustic oscillations will be the only possible form of motion. This is easy to prove when recalling that modes of discontinuous oscillations can be realized even in the case of $a = 0$ (free oscillations).

In view of the above, the frequency characteristic of a pipeline, designed with allowance for the possible existence of discontinuous oscillations, loses its unique character: At one and the same (sufficiently small) amplitude of piston oscillation, there may exist two substantially different oscillation modes in the system. The realization of any of the modes here depends on the history of the process. This feature of the frequency characteristic of the system represents one of the characteristic features of nonlinear systems, namely, the possibility of the existence of hysteresis.

2. Conditions of Existence of Periodic Solutions /36

Let us consider the steady state of forced discontinuous cavitational oscillations. Let us divide the oscillation period into two phases: the contact phase, during which the fluid contacts the piston, and the phase of free motion which occurs as a result of the cavitational discontinuity of the fluid due to reflection of the rarefaction wave from the piston. During the phase of free motion, the liquid is separated from the piston by the region of cavitation. The duration of the contact phase is $2L/c$ (this period is composed of the transit time of the hydraulic shock wave upward along the pipe to the tank, and the transit time of the rarefaction wave downward along the pipe to the piston). After this time has elapsed, the pressure at the piston drops discontinuously to the value of the saturated vapor pressure p_s, and the phase of free motion begins. (If, after reflection of the rarefaction wave from the piston, no cavitational discontinuity occurred, then the latter time would be characterized by a steep pressure drop.) The motion of the liquid during the contact phase will be defined by the following initial and boundary conditions:

$$p(0,t) = p_1(x), \quad u(x) = u_1(x), \quad \text{at} \quad t = 0 \tag{2.1}$$

$$u(0,t) = l_e^{\cdot}(t) = a\omega\cos(\omega t + \varphi), \quad \text{at} \quad x = 0 \tag{2.2}$$

$$p(L,t) = p_0 \quad \text{at} \quad x = L \tag{2.3}$$

Here, $p_1(x)$, $u_1(x)$ are the still unknown pressure and velocity distributions, along the liquid column at the beginning of the contact phase, and p_0 is the pressure in the tank. Equations (2.1), (2.2), (2.3), and (1.5) uniquely

determine the motion of the liquid during the contact phase and yield the velocity and pressure distribution along the liquid column at an arbitrary instant during the contact phase, in terms of their initial values. The required expressions in this case are easiest to derive by using the method of characteristics, which has been successfully employed by various authors for the study of problems connected with hydraulic shock (Bibl.6, 7). The relations (2.2) and (2.3), when using this method, define the conditions of reflection of the waves from the piston and tank. Using the method of characteristics, after the conventional constructions and calculations for problems of this type, the following relation between p and u at the beginning and end of the contact phase can be derived:

$$p_2(x) = -p_1(x) + p_0 + \rho c\left[l_e^{\cdot}\left(\frac{2L-x}{c}\right) - l_e^{\cdot}\left(\frac{x}{c}\right)\right] \tag{2.4}$$

$$u_2(x) = -u_1(x) + l_e^{\cdot}\left(\frac{2L-x}{c}\right) + l_e^{\cdot}\left(\frac{x}{c}\right) \tag{2.5}$$

where $p_2(x)$ and $u_2(x)$ are the velocity and pressure distributions along the liquid column at the end of the contact phase. The pressure of the fluid on the piston p_m is here defined by the relation

$$p_m(t) = \begin{cases} p_1(ct) - p_0 - \rho c u_1(ct) + \rho c l_e^{\cdot}(t) & \text{at } t < L/c \\ -p_1(2L-ct) + p_0 - \rho c u_1(2L-ct) + \rho c l_e^{\cdot}(t) & \text{at } t > L/c \end{cases} \tag{2.6}$$

Let us now pass to a description of the phase of free motion. The initial and boundary conditions defining the motion of the fluid in the phase of free motion, by virtue of the adopted hypotheses, will have the form

$$p(x) = p_2(x) + p_0, \qquad u(x) = u_2(x) \qquad \text{at } t = 0 \tag{2.7}$$

$$p(x) = p_s \quad \text{at} \quad x = 0, \qquad p(x) = p_0 \qquad \text{at } x = L \tag{2.8}$$

Here, p_0 is introduced to simplify the calculations; the initial reference time coincides with the beginning of the phase of free motion.

Let us reduce this problem to two simpler problems. Let us represent /37
the functions $p(x)$, $u(x)$, describing the pressure and velocity distribution along the liquid column during the phase of free motion, in the form of

$$p = M_p + N_p, \qquad \rho c u = M_u + N_u. \tag{2.9}$$

Here,

$$M_p = M_1(x - ct) + M_2(x + ct), \quad M_u = M_1(x - ct) - M_2(x + ct) \tag{2.10}$$

$$N_p = N_1(x - ct) + N_2(x + ct), \quad N_u = N_1(x - ct) + N_2(x + ct) \tag{2.11}$$

with the following initial and boundary conditions:

$$M_p = p_2(x), \quad M_u = \rho c u_2(x) \qquad \text{at } t = 0 \tag{2.12}$$

$$M_p = 0 \quad \text{at} \quad x = 0, \qquad M_p = 0 \quad \text{at } x = L \tag{2.13}$$

$$N_p = p_0, \quad N_u = 0 \quad \text{at } t = 0 \tag{2.14}$$

$$N_p = p_s \quad \text{at} \quad x = 0, \qquad N_p = p_0 \quad \text{at } x = L \tag{2.15}$$

It is not difficult to prove that the set of relations (2.9) - (2.15) describes the same problem as eqs.(1.5), (2.7), and (2.8).

Again making use of the method of characteristics, it is relatively easy to demonstrate that the functions $M_p(x, t)$ and $M_u(x, t)$ at the time $t = 2L/c$ have values coinciding with those of eqs.(2.12). It follows that the functions $M_p(x, t)$ and $M_u(x, t)$ are strictly periodic with a period equal to $2L/c$*. The functions M_p and M_u thus describe the free oscillations of the liquid column during the phase of free motion. The values of the functions M_u for $x = 0$ during the first oscillation period of the fluid in the phase of free motion, are thus found to be

$$M_u(0,t) = \begin{cases} -p_2 + \rho c u_2(ct) & \text{at} \quad 0 < t < L/c \\ \rho c u_2(t) & \text{at} \quad t = L/c \\ p_2(2L - ct) + \rho c u_2(2L - ct) & \text{at} \quad L/c < t < 2L/c \end{cases} \tag{2.16}$$

Since the functions M_u are periodic, this relation repeats during the entire phase of free motion. As for the function N_u, the solution for time intervals $2L/c$ obtainable by the method of characteristics shows that its value holds constant but, at the end of these time intervals, undergoes discontinuous changes equal to $2(p_0-p_s)/\rho c$. The value of the function N_u at $x = 0$ during the time interval $2(n - 1)\ L/c < t < 2nL/c$, where n is the number of oscillations of the function N_u, counted from the beginning of the phase of free motion, will then be

$$N_u = -(2n - 1)\ \Delta p \qquad (\Delta p = p_0 - p_s). \tag{2.17}$$

It is of substantial importance for our further discussion that the above time intervals are open, so that the value of the function for instants of time that are multiples of $2L/c$ is not determined by eq.(2.17). The value of the function N_u at this instant, as shown by the method of characteristics, will be

$$N_u(x, 2nL/c) = -2n\Delta p. \tag{2.18}$$

Let us now calculate the coordinate of the lower end of the liquid column at the time T_n, defined by the relation

$$T_n = 2n\tau \quad (\tau = L/c). \tag{2.19}$$

To these times there corresponds an integer of complete periods of free /38
oscillation of the liquid in the phase of free motion. Denoting the required value of the coordinate by l_n, we can then write

$$l_n = \int_0^{2n\tau} u(0, t)\, dt. \tag{2.20}$$

Substituting the value of u obtained from eq.(2.9) into eq.(2.20), and bearing in mind that the function M_u has the period $2L/c$ and that the function

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* As was to be expected, starting out from the form of the boundary conditions [eq.(2.13)], this period coincides with the period of the harmonic of the free oscillations of a liquid column open at both ends.

N_u, at x = 0, is defined by eq.(2.17), we obtain

$$l_n = \frac{1}{\rho c}\int_0^{2n\tau} M_u(0,\ t)\,dt + \frac{1}{\rho c}\int_0^{2n\tau} N_u(0,\ t)dt = \frac{n}{\rho c}\int_0^{2\tau} M_u(0,\ t)\,dt - 2\frac{L}{c}\frac{\Delta p}{\rho c}\sum_{n'=1}^{n}(2n'-1) = -2n^2\frac{\Delta p}{\rho c^2} + \frac{n}{\rho c}\int_0^{2\tau} M_u(0,\ t)\,dt \qquad (2.21)$$

The integral of M_u, making use of eq.(2.16), can be rewritten in the form of

$$I = \int_0^{2\tau} M_u(0,\ t)\,dt = -\int_0^{\tau} p_2(ct)\,dt + \rho c\int_0^{\tau} u_2(ct)\,dt + \int_{\tau}^{2\tau} p_2(2L-ct)\,dt + \rho c\int_{\tau}^{2\tau} u_2(2L-ct)\,dt\ . \qquad (2.22)$$

Passing now to new integration variables (for the first two integrals, ct; for the last two integrals, 2L - ct) and making simple transformations, we obtain

$$I = 2\rho\int_0^{L} u_2(z)\,dz\ . \qquad (2.23)$$

Substituting eq.(2.23) into eq.(2.21) will yield

$$l_n = 2\frac{L}{c}n\left(u_0 - n\frac{\Delta p}{\rho c}\right) \qquad (2.24)$$

where u_0 is the mean velocity at the beginning of the phase of free motion, averaged along the liquid column,

$$u_0 = \frac{1}{L}\int_0^{L} u_2(x)\,dx\ . \qquad (2.25)$$

To formulate the conditions of periodicity, the function N_p must still be determined. The method of characteristics shows that the function N_p, of a frequency of c/2L assumes a value constant along the length:

$$N_p = p_0\ . \qquad (2.26)$$

Let us now write the existence conditions of periodic solutions. The period of oscillation of the fluid in the phase of free motion is a multiple of 2L/c while the contact time is also equal to 2L/c, so that the period of strictly periodic solutions of the discontinuous oscillations must satisfy the condition:

$$T = \frac{2\pi m}{\omega} = 2n_0\frac{L}{c} + 2\frac{L}{c} = T_0 + \frac{2L}{c} \qquad (2.27)$$

Here m and n_0 are integers, with the first showing how many times the oscillation period of the fluid is greater than the oscillation period of the piston (of the order of fractional resonance), and the second showing the number /39

of oscillation periods of the liquid column during the phase of free motion; T_0 is the duration of the phase of free motion.

In addition to this: a) The velocity and pressure distribution along the liquid column at the end of the phase of free motion and at the beginning of the contact phase must coincide. According to the above condition of compatibility of the solutions, to eqs.(2.9), (2.12), and (2.14), and to the periodicity of the functions M_p and N_p, as just established, the distribution of pressure at the end of the phase of free motion is defined by the relation

$$p_1(x) = M_p\left(x,\ 2n_0\frac{L}{c}\right) + N_p\left(x,\ 2n_0\frac{L}{c}\right) = p_2(x) + p_0 \tag{2.28}$$

Similarly, making use of the existence condition of the solutions, of the periodicity of the function M_u, and of eqs.(2.9), (2.12), and (2.14), we obtain

$$u_1(x) = M_u\left(x,\ 2n_0\frac{L}{c}\right) + N_u\left(x,\ 2n_0\frac{L}{c}\right) = u_2(x) - 2n_0\frac{\Delta p}{\rho c} \tag{2.29}$$

b) Since, at the time $t = 2L/c$ (the beginning of the phase of free motion) and at $t = T_0 + 2L/c$ (the end of the phase of free motion*), the coordinate of the lower end of the liquid column coincides with the coordinate of the piston, it follows that the relation

$$l_e\left(T_0 + 2\frac{L}{c}\right) = l_e\left(2\frac{L}{c}\right) + \int_0^{T_0} u\ (0,\ t)\ dt = l_e\left(2\frac{L}{c}\right) + l_n, \tag{2.30}$$

must be satisfied.

Equations (2.4), (2.5), (2.27), (2.28), (2.29), (2.30) are the existence conditions for strictly periodic solutions.

3. Amplitudes, Phases and Forms of Strictly Periodic Solutions

Solving eqs.(2.4), (2.5), (2.28), and (2.29) for the functions $p_1(x)$, $u_1(x)$, $p_2(x)$, $u_2(x)$, and making use of eqs.(1.1), we obtain, after simple transformations,

$$p_1(x) = p_0 - \rho c a\omega \sin(\omega L/c + \varphi)\ \sin[\omega(L - x)/c] \tag{3.1}$$

$$u_1(x) = -n_0\Delta p/\rho c + a\omega\cos(\omega L/c + \varphi)\cos[\omega(L - x)/c] \tag{3.2}$$

$$p_2(x) = \rho c a\omega \sin(\omega L/c + \varphi)\sin[\omega(L - x)/c] \tag{3.3}$$

$$u_2(x) = n_0\Delta p/\rho c + a\omega\cos(\omega L/c + \varphi)\cos\omega(L - x)/c \tag{3.4}$$

In these relations, one unknown quantity still remains, namely, the phase shift φ of the piston oscillations relative to the beginning of the contact phase. To determine this quantity, let us make use of eq.(2.30). Substituting eqs.(1.1) and (2.24) into this formula and utilizing eqs.(2.25) and (3.4), we obtain

$$(1 + n_0)\cos(\omega L/c + \varphi)\sin(\omega L/c) = 0 \tag{3.5}$$

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* The same initial time is here prescribed for both phases of motion.

Further, on the basis of eq.(2.27), we find

$$\cos\left(\frac{\pi m}{n_0+1}+\varphi\right)\sin\frac{\pi m}{n_0+1}=0\ . \qquad (3.6)$$

It can be demonstrated that the fractional resonances of high order are not attained in the presence of energy dissipation and that the solutions obtained here can be effectively applied only at rather high values of n_0 so that, in eq.(3.6), we can set $m \leq 3$ and $n_0 > 2$. When these conditions are satisfied, $\sin [\pi m/(1 + n_0)] \neq 0$.

Equating the first factor of eq.(3.6) to zero, we obtain /40

$$\Phi_1=\varphi_1+\omega L/c={}^1/_2\pi, \qquad \Phi_2=\varphi_2+\omega L/c={}^3/_2\pi\ . \qquad (3.7)$$

Here Φ_1 is the phase shift between the piston oscillations and the middle of the contact phase. To the two values of the phase Φ there correspond two

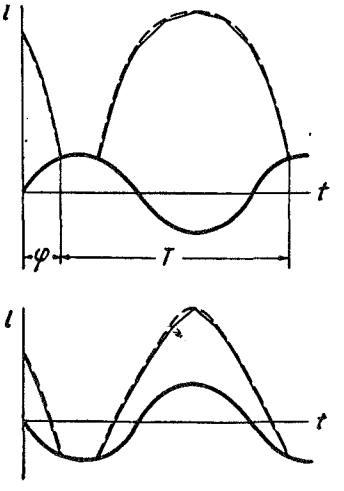

Fig.3

different oscillation modes. On realization of the first oscillation mode (with the phase angle Φ_1), the contact phase enters the region of top center of the

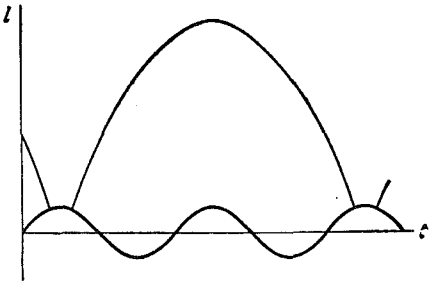

Fig.4

piston, while on realization of the second oscillation mode, it enters the region of bottom center. A qualitative analysis of the oscillation stability, making use of the dependence of the energy increment on the value of the phase angle φ, shows that in the range of sufficiently low frequencies, only the first oscillation mode is stable. We expect to report on a more rigorous analysis of oscillation stability in a separate paper. The heavy line in Fig.3 represents the piston coordinate and the thin line, the boundaries of the cavitation region ($m = 1$). The upper portion of the diagram corresponds to the first oscillation mode and the lower, to the second mode*. Figure 4 shows the analogous picture for the fractional resonance, at $m = 2$.

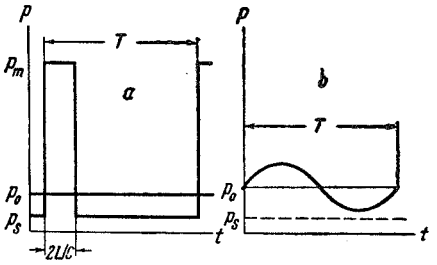

Fig.5

Substituting eq.(3.7) into eqs.(3.2) and (3.4), and bearing in mind eq.(2.25), we obtain

$$u_2(x) = -u(x) = n_0 \frac{\Delta p}{\rho c} = u_0 . \tag{3.8}$$

To find the value of the pressure exerted on the piston during the contact phase, it is sufficient to make use of eqs.(2.6), (3.1), (3.2), (3.8), (1.1), and (3.7); after transformations, this yields

$$p_m = p_0 + n_0 \Delta p = p_0 + \rho c u_0 . \tag{3.9}$$

It follows from eq.(3.9) that the pressure exerted on the piston during the contact phase is independent of time and has the same value for both oscillation modes. Its absolute value here is equal to the impact pressure created on sudden stoppage of the liquid and having a velocity of u_0 and a pressure of p_0. According to our results, the time dependence of the pressure exerted on the piston will have the form shown in Fig.5a.

Substituting into eq.(3.9) the value of n_0 obtained from eq.(2.27), it /41 is easy to obtain the following relation between μ, the dimensionless amplitude of the pressure oscillations, and the frequency:

$$\mu = \frac{p_m - p_s}{p_0 - p_s} = \frac{m}{2f} \frac{c}{L} = \frac{m}{2f^*} \tag{3.10}$$

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* The same diagrams also give a comparison of the exact and approximate solutions of the problem obtained in the following Section.

The frequency dependence of the dimensionless oscillation amplitude for the modes of discontinuous oscillations at m = 1 and m = 2, given by eq.(3.10), is shown in Fig.6 by black dots.

4. Approximate Model of the Phenomenon and Discussion of the Results

Although eq.(3.10) does not contain n_0 in explicit form, it should not be forgotten that it is valid only for integral values of n_0. If this condition is not satisfied, then, using the method of characteristics, it can be shown that the velocity and pressure distribution along the liquid column at the time corresponding to the beginning of two successive contact phases will not coincide. The motion of the fluid then ceases to be strictly periodic. Both the amplitude of pressure oscillations* and the oscillation period will in this case vary periodically (or quasi-periodically).

On the other hand, eq.(3.10) permits an interpolation of the frequency characteristics of the system for values of the frequency that do not satisfy the condition of n_0 being an integer. The corresponding relation is shown in Fig.6 by the heavy line.

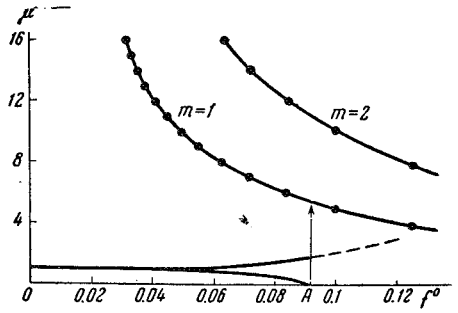

Fig.6

In connection with our result, this raises the question as to whether eq.(3.10) can be used at frequency values that do not satisfy the condition of integrality of n_0. Assuming that eq.(3.10), in a certain sense, describes the approximate frequency dependence of the amplitude at any oscillation frequency and comparing the approximate relation with the points corresponding to the exact relation, it is easy to prove that, in the region of sufficiently great n_0, the points corresponding to the exact solutions are closely packed (Fig.6). This permits the postulate that the divergence between the exact and approximate solutions is slight in the region of low frequencies and high amplitudes of the discontinuous oscillations, while the initial relations admit of some simplification in this region.

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* In this case, the time dependence of the pressure during the contact phase loses the simple form shown in Fig.5.

To determine the validity of this hypothesis, let us turn to the relations describing the motion of a liquid in the phase of free motion.

According to eqs.(2.9), (2.16), (2.18), and (3.8), the velocity at the lower end of the liquid column, at the time $T_n = 2nL/c$, will be

$$u_n = u_0 - 2n\frac{\Delta p}{\rho c}. \tag{4.1}$$

Making use of eq.(2.19), eqs.(4.1) and (2.24) can be transformed into the following form:

$$\begin{aligned} u_n &= u_0 - \frac{\Delta p}{\rho L} T_n \\ l_n &= u_0 T_n - \frac{1}{2}\frac{\Delta p}{\rho L} T_n^2 \end{aligned} \tag{4.2}$$

On changing from the discrete time T_n to the continuous time (t), it is easy to demonstrate that eqs.(4.2) coincide with the solution of a differential equation describing the motion of a column of incompressible fluid under the action of the pressure drop Δp:

$$\rho L \frac{d^2 l}{dt^2} = -\Delta p \tag{4.3}$$

under the initial conditions t = 0 (beginning of the phase of free motion), $l = 0$ and $\dot{l} = u_0$.

These relations show that the motion of the lower ends of the column of liquid may be represented in the form of the components: translational velocity u_0^*, and relative velocity, due to alternating compression and expansion of the liquid column. The oscillation frequency of the liquid column in its relative motion is c/2L, while the frequency of the forced oscillations is $(m/n_0 + 1)(L/2c)$. In the region of sufficiently great n_0, to which correspond discontinuous oscillations of higher amplitude, the relative velocity thus plays the role of a high-frequency oscillation component. It follows that, to describe the discontinuous cavitational oscillations in first approximation, without taking account of the high-frequency components, the wave processes in the phase of free motion may be neglected, and to describe this phase we may use the equation of motion of an incompressible fluid [eq.(4.3)]. If the condition (2.27) is satisfied, the resultant solutions will coincide with the exact solutions. At any other frequency, however, the time dependence of the pressure shown in Fig.5 will be distorted by the presence of a high-frequency component. Making use of the relation of the preceding Section, we can show that, in the region of sufficiently low frequencies (high values of n_0), the oscillation amplitude of the high-frequency component becomes substantially less than the oscillation amplitude defined by eq.(3.10). We will not present the proof here, since it is of limited interest (under actual conditions, dissipative processes lead to a substantial decrease in the amplitude of the high-frequency component).

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* We recall that u_0 is the velocity of the liquid, averaged along the column, at the beginning of the phase of free motion.

Equations (4.3) and (3.10) can be used in a frequency range not satisfying the condition of integrality of n_0 only if the values of n_0 are sufficiently high (low frequencies). Since, on the one hand, at low values of n_0 the amplitude of the hydraulic shock, according to eq.(3.10), decreases to values that no longer ensure the appearance of cavitational discontinuities after reflection of the return wave from the piston, this restriction is not rigorous.

In conclusion, let us consider in more detail the frequency characteristic of the system under examination, taking account of the possibility of realizing the discontinuous oscillation mode.

Figure 6 shows the frequency characteristic corresponding to the acoustic theory [eq.(1.7)] in the form of two lines, with the upper curve corresponding to the maximum deviation of pressure and the lower, to the minimum deviation. At the point A, the minimum pressure reaches a value of p_s, and the mode of acoustic oscillations becomes unstable. If the frequency of the piston oscillations is slowly increased, then up to the point A the amplitude of the pressure oscillations of the fluid will monotonically increase with the frequency. At the point A, the acoustic oscillation mode breaks off and discontinuous oscillations are set up in the system. The oscillations and pressure at the pistons now lose their sinusoidal character and assume the forms plotted in Fig.5a. On further increase in the oscillation frequency, the oscillation amplitude will decrease.

If after decay of the acoustic oscillation mode, at the point A, the oscillation frequency of the piston declines, the system will not return to the acoustic oscillation mode. The amplitude of the pressure oscillations will now monotonically increase with decreasing frequency. As f^0 approaches zero, the oscillation amplitude, according to eq.(3.10), increases without limit. In real systems, the existence of energy dissipation must necessarily lead to a limitation of the amplitudes in this frequency range.

The resultant phenomenon of hysteresis is characteristic and is well known for a rather large class of nonlinear systems. A comparison of the frequency characteristics corresponding to the acoustic and discontinuous oscillations shows substantial differences. The frequency characteristic of the discontinuous oscillations has a rather large number of purely nonlinear properties. These properties include the absence of any dependence of this oscillation amplitude on the oscillation amplitude of the piston, and also the possibility of realizing fractional resonances ($m \neq 1$).

Another striking fact is the directly opposite influence of frequency on the amplitude of the acoustic and discontinuous oscillations: (the acoustic oscillation amplitude decreases with decreasing frequency while the amplitude of the discontinuous oscillation mode increases). Further than that, as follows from the definition of the quantity μ, the absolute value of the discontinuous oscillation amplitude depends on the pressure in the tank (p_0), which is not the case for the acoustic oscillations.

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